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1997 J. Phys. A: Math. Gen. 30 8325

(http://iopscience.iop.org/0305-4470/30/23/028)

J. Phys. A: Math. Gen. **30** (1997) 8325–8335. Printed in the UK PII: S0305-4470(97)86115-X

Comments on the Drinfeld realization of the quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$ and its Hopf algebra structure

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Received 21 July 1997

Abstract. By generalizing the Reshetikhin and Semenov-Tian-Shansky construction to supersymmetric cases, we obtain the Drinfeld current realization for the quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$. We find a simple coproduct for the quantum current generators and establish the Hopf algebra structure of this super current algebra.

1. Introduction

This paper contains two results on the Drinfeld second realization (current realization) [1] for the untwisted quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$.

The first result is the extension of the Reshetikhin and Semenov-Tian-Shansky (RS) construction [2] to supersymmetric cases. Using this super RS construction and a super version of the Ding–Frenkel theorem [3], we obtain the defining relations for $U_q[gl(m|n)^{(1)}]$ in terms of super (or graded) current generators (generating functions). In [4], the authors made a similar effort. However, the relations they obtained are *not* supersymmetric and the algebra they defined is not a current realization of $U_q[gl(m|n)^{(1)}]$ but rather a current realization of a 'non-standard' quantum *bosonic* algebra associated with $U_q[gl(N = m+n)]$. This is because those authors failed to take care of the *grading* in the multiplication rule of tensor products which plays a fundamental role in any supersymmetric theory.

The second result is the coproduct, counit and antipode for the current realization of $U_q[gl(m|n)^{(1)}]$, thus establishing a Hopf algebra structure of this super current algebra.

2. The super RS algebra and the Ding–Frenkel theorem

Let us start with introducing some useful notation. The graded Yang–Baxter equation (YBE) with spectral-parameter dependence takes the form

$$R_{12}\left(\frac{z}{w}\right)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}\left(\frac{z}{w}\right) \tag{2.1}$$

where $R(z) \in End(V \otimes V)$, V being a graded vector space, and obeys the weight conservation condition: $R(z)_{\alpha\beta}^{\alpha'\beta'} \neq 0$ only when $[\alpha'] + [\beta'] + [\alpha] + [\beta] = 0 \mod 2$.

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The multiplication rule for the tensor product is defined for the homogeneous elements $a,\ b,\ c,\ d$ of a quantum superalgebra by

$$(a\otimes b)(c\otimes d)=(-1)^{[b][c]}\,(ac\otimes bd) \tag{2.2}$$

where $[a]\in\mathbb{Z}_2$ denotes the grading of the element a.

We introduce the graded permutation operator P on the tensor product module $V\otimes V$ such that $P(v_\alpha\otimes v_\beta)=(-1)^{[\alpha][\beta]}(v_\beta\otimes v_\alpha)\,,\ \forall v_\alpha,v_\beta\in V$. In most cases the R-matrix possesses, among others, the following properties:

$$\text{(i)}\qquad P_{12}R_{12}(z)P_{12}=R_{21}(z) \tag{2.3}$$

$$\text{(ii)}\qquad R_{12}\left(\frac{z}{w}\right)R_{21}\left(\frac{w}{z}\right)=1. \tag{2.4}$$

The graded YBE, when written in matrix form, carries extra signs, and depending on the definition of the matrix elements, these sign factors differ [5]. If we define the matrix elements of $R(z)$ by

$$R(z)(v^{\alpha'}\otimes v^{\beta'})=R(z)^{\alpha'\beta'}_{\alpha\beta}(v^\alpha\otimes v^\beta) \tag{2.5}$$

then the matrix YBE [6] reads

$$\begin{aligned}R\left(\frac{z}{w}\right)^{\alpha'\beta'}_{\alpha\beta}R(z)^{\alpha''\gamma'}_{\alpha'\gamma}R(w)^{\beta''\gamma''}_{\beta'\gamma'}(-1)^{[\alpha][\beta]+[\gamma][\alpha']+[\gamma'][\beta']}\\ =R(w)^{\beta'\gamma'}_{\beta\gamma}R(z)^{\alpha'\gamma''}_{\alpha\gamma'}R\left(\frac{z}{w}\right)^{\alpha''\beta''}_{\alpha'\beta'}(-1)^{[\beta][\gamma]+[\gamma'][\alpha]+[\beta'][\alpha']}.\end{aligned} \tag{2.6}$$

After the redefinition

$$\tilde{R}(z)^{\alpha'\beta'}_{\alpha\beta}=R(z)^{\alpha'\beta'}_{\alpha\beta}\,(-1)^{[\alpha][\beta]} \tag{2.7}$$

the extra signs in (2.6) disappear. However, this redefinition does not preserve its semiclassical properties.

In matrix form the graded permutation operator $P=\sum_{\alpha,\beta}(-1)^{[\beta]}\,E^\alpha_\beta\otimes E^\beta_\alpha$ reads

$$P^{\alpha'\beta'}_{\alpha\beta}=\delta_{\alpha\beta'}\delta_{\alpha'\beta}(-1)^{[\alpha'][\beta']}. \tag{2.8}$$

The RS construction [2] can be generalized to supersymmetric cases. Formally, the super RS algebra is defined by similar relations as in non-supersymmetric cases [2, 3], but tensor products now carry gradings. We are thus led to the following definition.

Definition 1. The super RS algebra is generated by invertible $L^\pm(z)$ which satisfy

$$\begin{aligned}R\left(\frac{z}{w}\right)L_1^\pm(z)L_2^\pm(w)&=L_2^\pm(w)L_1^\pm(z)R\left(\frac{z}{w}\right)\\ R\left(\frac{z_+}{w_-}\right)L_1^+(z)L_2^-(w)&=L_2^-(w)L_1^+(z)R\left(\frac{z_-}{w_+}\right)\end{aligned} \tag{2.9}$$

where $L_1^\pm(z)=L^\pm(z)\otimes 1,\ L_2^\pm(z)=1\otimes L^\pm(z)$ and $z_\pm=zq^{\pm\frac{1}{2}c}$. In the first expression of (2.9) the expansion direction of $R(\frac{z}{w})$ can be chosen in $\frac{z}{w}$ or $\frac{w}{z}$, but in the second the expansion direction must only be in $\frac{z}{w}$.

The super RS algebra is a Hopf algebra: its coproduct is defined by

$$\Delta(L^{\pm}(z) = L^{\pm}(zq^{\pm 1\otimes\frac{1}{2}c}) \dot{\otimes} L^{\pm}(zq^{\mp\frac{1}{2}c\otimes 1}) \tag{2.10}$$

and its antipode is

$$S(L^{\pm}(z)) = L^{\pm}(z)^{-1}. \tag{2.11}$$

In matrix form, equation (2.9) carries extra signs due to the graded multiplication rule of tensor products:

$$\begin{aligned}
&R\left(\frac{z}{w}\right)_{\alpha\beta}^{\alpha''\beta''} L^{\pm}(z)_{\alpha''}^{\alpha'} L^{\pm}(w)_{\beta''}^{\beta'} (-1)^{[\alpha']([\beta']+[\beta''])} \\
&\qquad = L^{\pm}(w)_{\beta}^{\beta''} L^{\pm}(z)_{\alpha}^{\alpha''} R\left(\frac{z}{w}\right)_{\alpha''\beta''}^{\alpha'\beta'} (-1)^{[\alpha]([\beta]+[\beta''])} \\
&R\left(\frac{z_+}{w_-}\right)_{\alpha\beta}^{\alpha''\beta''} L^{+}(z)_{\alpha''}^{\alpha'} L^{-}(w)_{\beta''}^{\beta'} (-1)^{[\alpha']([\beta']+[\beta''])} \\
&\qquad = L^{-}(w)_{\beta}^{\beta''} L^{+}(z)_{\alpha}^{\alpha''} R\left(\frac{z_-}{w_+}\right)_{\alpha''\beta''}^{\alpha'\beta'} (-1)^{[\alpha]([\beta]+[\beta''])}.
\end{aligned} \tag{2.12}$$

We introduce the matrix θ:

$$\theta_{\alpha\beta}^{\alpha'\beta'} = (-1)^{[\alpha][\beta]} \delta_{\alpha\alpha'} \delta_{\beta\beta'}. \tag{2.13}$$

With the help of this matrix θ, one can cast (2.12) as the usual matrix equation:

$$\begin{aligned}
&R\left(\frac{z}{w}\right) L_1^{\pm}(z)\theta L_2^{\pm}(w)\theta = \theta L_2^{\pm}(w)\theta L_1^{\pm}(z) R\left(\frac{z}{w}\right) \\
&R\left(\frac{z_+}{w_-}\right) L_1^{+}(z)\theta L_2^{-}(w)\theta = \theta L_2^{-}(w)\theta L_1^{+}(z) R\left(\frac{z_-}{w_+}\right).
\end{aligned} \tag{2.14}$$

Now the multiplications in (2.14) are simply the usual matrix multiplications.

In this paper we take $R(\frac{z}{w}) \in End(V \otimes V)$ to be the R-matrix associated with $U_q[gl(m|n)]$, where V is a $(m+n)$-dimensional graded vector space. Let the basis vectors $\{v^1, v^2, \ldots, v^m\}$ be even and $\{v^{m+1}, v^{m+2}, \ldots, v^{m+n}\}$ be odd. Then the R-matrix has the following matrix elements:

$$\begin{aligned}
&R\left(\frac{z}{w}\right)_{\alpha\beta}^{\alpha'\beta'} = (-1)^{[\alpha][\beta]} \tilde{R}\left(\frac{z}{w}\right)_{\alpha\beta}^{\alpha'\beta'} \\
&\tilde{R}\left(\frac{z}{w}\right) = \sum_{i=1}^{m} E_i^i \otimes E_i^i + \sum_{i=m+1}^{m+n} \frac{wq - zq^{-1}}{zq - wq^{-1}} E_i^i \otimes E_i^i + \frac{z-w}{zq - wq^{-1}} \sum_{i\neq j} (-1)^{[i][j]} E_i^i \otimes E_j^j \\
&\qquad \times \sum_{i<j} \frac{z(q-q^{-1})}{zq - wq^{-1}} E_i^j \otimes E_j^i + \sum_{i>j} \frac{w(q-q^{-1})}{zq - wq^{-1}} E_i^j \otimes E_j^i.
\end{aligned} \tag{2.15}$$

It is easy to check that the R-matrix $R(z)$ satisfies equations (2.3) and (2.4). We will construct the Drinfeld realization of $U_q[gl(m|n)^{(1)}]$. We first state a super version of the Ding–Frenkel theorem [3].

Theorem 1. $L^{\pm}(z)$ has the following unique Gauss decomposition

$$L^{\pm}(z) = \begin{pmatrix} 1 & \cdots & & 0 \\ e^{\pm}_{2,1}(z) & \ddots & & \\ e^{\pm}_{3,1}(z) & & & \vdots \\ \vdots & & & \\ e^{\pm}_{m+n,1}(z) & \cdots & e^{\pm}_{m+n,m+n-1}(z) & 1 \end{pmatrix} \begin{pmatrix} k^{\pm}_1(z) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & k^{\pm}_{m+n}(z) \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & f^{\pm}_{1,2}(z) & f^{\pm}_{1,3}(z) & \cdots & f^{\pm}_{1,m+n}(z) \\ \vdots & \ddots & \cdots & & \vdots \\ & & & & f^{\pm}_{m+n-1,m+n}(z) \\ 0 & & & & 1 \end{pmatrix} \tag{2.16}$$

where $e^{\pm}_{i,j}(z)$, $f^{\pm}_{j,i}(z)$ and $k^{\pm}_i(z)$ $(i > j)$ are elements in the super RS algebra and $k^{\pm}_i(z)$ are invertible. Let

$$\begin{aligned} X^{-}_i(z) &= f^{+}_{i,i+1}(z_+) - f^{-}_{i,i+1}(z_-) \\ X^{+}_i(z) &= e^{+}_{i+1,i}(z_-) - e^{-}_{i+1,i}(z_+) \end{aligned} \tag{2.17}$$

where $z_{\pm} = zq^{\pm\frac{1}{2}c}$, then $q^{\pm\frac{1}{2}c}$, $X^{\pm}_i(z)$, $k^{\pm}_j(z)$, $i = 1, 2, \ldots, m+n-1$, $j = 1, 2, \ldots, m+n$ give the defining relations of quantum affine superalgebra $U_q[gl(m|n)^{(1)}]$.

The Gauss decomposition implies that the elements $e^{\pm}_{i,j}(z)$, $f^{\pm}_{j,i}(z)$ $(i > j)$ and $k^{\pm}_i(z)$ are uniquely determined by $L^{\pm}(z)$. In what follows we will denote $f^{\pm}_{i,i+1}(z)$, $e^{\pm}_{i+1,i}(z)$ as $f^{\pm}_i(z)$, $e^{\pm}_i(z)$, respectively.

The following matrix equations can be deduced from equations (2.14):

$$R_{21}\left(\frac{z}{w}\right)\theta L^{\pm}_2(z)\theta L^{\pm}_1(w) = L^{\pm}_1(w)\theta L^{\pm}_2(z)\theta R_{21}\left(\frac{z}{w}\right) \tag{2.18}$$

$$R_{21}\left(\frac{z_-}{w_+}\right)\theta L^{-}_2(z)\theta L^{+}_1(w) = L^{+}_1(w)\theta L^{-}_2(z)\theta R_{21}\left(\frac{z_+}{w_-}\right) \tag{2.19}$$

$$\theta L^{\pm}_2(z)^{-1}\theta L^{\pm}_1(w)^{-1}R_{21}\left(\frac{z}{w}\right) = R_{21}\left(\frac{z}{w}\right)L^{\pm}_1(w)^{-1}\theta L^{\pm}_2(z)^{-1}\theta \tag{2.20}$$

$$\theta L^{+}_2(z)^{-1}\theta L^{-}_1(w)^{-1}R_{21}\left(\frac{z_+}{w_-}\right) = R_{21}\left(\frac{z_-}{w_+}\right)L^{-}_1(w)^{-1}\theta L^{+}_2(z)^{-1}\theta \tag{2.21}$$

$$L^{\pm}_1(w)^{-1}R_{21}\left(\frac{z}{w}\right)\theta L^{\pm}_2(z)\theta = \theta L^{\pm}_2(z)\theta R_{21}\left(\frac{z}{w}\right)L^{\pm}_1(w)^{-1} \tag{2.22}$$

$$L^{-}_1(w)^{-1}R_{21}\left(\frac{z_+}{w_-}\right)\theta L^{+}_2(z)\theta = \theta L^{+}_2(z)\theta R_{21}\left(\frac{z_-}{w_+}\right)L^{-}_1(w)^{-1} \tag{2.23}$$

$$L^{+}_1(w)^{-1}R_{21}\left(\frac{z_-}{w_+}\right)\theta L^{-}_2(z)\theta = \theta L^{-}_2(z)\theta R_{21}\left(\frac{z_+}{w_-}\right)L^{+}_1(w)^{-1} \tag{2.24}$$

where $R_{21}(\frac{z}{w}) = R(\frac{w}{z})^{-1}$. As in equations (2.14), the multiplications in (2.18)–(2.24) are usual matrix multiplications.

3. The $m = 1,\ n = 1$ case: $U_q[gl(1|1)^{(1)}]$

For the simplest supersymmetric case $U_q[gl(1|1)^{(1)}]$, the $L^{\pm}(z)$ take the form

$$L^{\pm}(z) = \begin{pmatrix} k_1^{\pm}(z) & k_1^{\pm}(z) f_1^{\pm}(z) \\ e_1^{\pm}(z)k_1^{\pm}(z) & k_2^{\pm}(z) + e_1^{\pm}(z)k_1^{\pm}(z) f_1^{\pm}(z) \end{pmatrix}. \tag{3.1}$$

The R-matrix and θ are given by

$$R\left(\frac{z}{w}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{z-w}{zq-wq^{-1}} & \dfrac{z(q-q^{-1})}{zq-wq^{-1}} & 0 \\ 0 & \dfrac{w(q-q^{-1})}{zq-wq^{-1}} & \dfrac{z-w}{zq-wq^{-1}} & 0 \\ 0 & 0 & 0 & -\dfrac{wq-zq^{-1}}{zq-wq^{-1}} \end{pmatrix} \tag{3.2}$$

and $\theta = \mathrm{diag}(1, 1, 1, -1)$, respectively.

Using equations (2.13), (2.15) and (2.14), (2.18)–(2.24), and by calculations similar to those for the non-super case [3], we obtain a super RS algebra, which, by means of theorem 1, leads to the following definition.

Definition 2. $U_q[gl(1|1)^{(1)}]$ is an associative algebra with unit 1 and the Drinfeld current generators $X_1^{\pm}(z)$, $k_i^{\pm}(z)$ $(i = 1, 2)$, a central element c and a non-zero complex parameter q. $k_i^{\pm}(z)$ are invertible. The gradings of the generators are $[X_1^{\pm}(z)] = 1$ and $[k_i^{\pm}(z)] = 0 = [c]$. The relations read

$$\begin{aligned}
&k_i^{\pm}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\pm}(z) \qquad i,\ j = 1, 2 \\
&k_1^{+}(z)k_1^{-}(w) = k_1^{-}(w)k_1^{+}(z) \\
&\frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}}k_2^{+}(z)k_2^{-}(w) = \frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}}k_2^{-}(w)k_2^{+}(z) \\
&\frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}k_2^{\mp}(w)^{-1}k_1^{\pm}(z) = \frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}k_1^{\pm}(z)k_2^{\mp}(w)^{-1} \\
&k_i^{\pm}(z)^{-1}X_1^{-}(w)k_i^{\pm}(z) = \frac{z_\mp q - wq^{-1}}{z_\mp - w}X_1^{-}(w) \\
&k_i^{\pm}(z)X_1^{+}(w)k_i^{\pm}(z)^{-1} = \frac{z_\pm q - wq^{-1}}{z_\pm - w}X_1^{+}(w) \\
&\{X_1^{\pm}(z), X_1^{\pm}(w)\} = 0 \\
&\{X_1^{+}(z), X_1^{-}(w)\} = (q-q^{-1})\left(\delta\left(\frac{w}{z}q^{c}\right)k_2^{+}(w_+)k_1^{+}(w_+)^{-1}\right. \\
&\qquad\qquad \left. -\,\delta\left(\frac{w}{z}q^{-c}\right)k_2^{-}(z_+)k_1^{-}(z_+)^{-1}\right)
\end{aligned} \tag{3.3}$$

where $\{X, Y\} \equiv XY + YX$ denotes an anti-commutator and

$$\delta(z) = \sum_{k\in\mathbb{Z}} z^k \tag{3.4}$$

is a formal series.

Theorem 2. The algebra $U_q[gl(1|1)^{(1)}]$ defined by (3.3) has a Hopf algebra structure, which is given by the following formulae.

Coproduct Δ:

$$\begin{aligned}
&\Delta(q^c) = q^c \otimes q^c \\
&\Delta(k_i^+(z)) = k_i^+(zq^{\frac{1}{2}c_2}) \otimes k_i^+(zq^{-\frac{1}{2}c_1}) \\
&\Delta(k_i^-(z)) = k_i^-(zq^{-\frac{1}{2}c_2}) \otimes k_i^-(zq^{\frac{1}{2}c_1}) \\
&\Delta(X_1^+(z)) = X_1^+(z) \otimes 1 + \psi_1(zq^{\frac{1}{2}c_1}) \otimes X_1^+(zq^{c_1}) \\
&\Delta(X_1^-(z)) = 1 \otimes X_1^-(z) + X_1^-(zq^{c_2}) \otimes \phi_1(zq^{\frac{1}{2}c_2})
\end{aligned} \tag{3.5}$$

where $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$, $\psi_1(z) = k_2^-(z)k_1^-(z)^{-1}$ and $\phi_1(z) = k_2^+(z)k_1^+(z)^{-1}$.

Counit ϵ:

$$\epsilon(q^c) = 1 \qquad \epsilon(k_i^\pm(z)) = 1 \qquad \epsilon(X_1^\pm(z)) = 0. \tag{3.6}$$

Antipode S:

$$\begin{aligned}
&S(q^c) = q^{-c} \qquad S(k_i^\pm(z)) = k_i^\pm(z)^{-1} \quad i = 1, 2 \\
&S(X_1^+(z)) = -\psi_1(zq^{-\frac{1}{2}c})^{-1} X_1^+(zq^{-c}) \\
&S(X_1^-(z)) = -X_1^-(zq^{-c})\phi_1(zq^{-\frac{1}{2}c})^{-1}.
\end{aligned} \tag{3.7}$$

Proof. The proof is rather elementary. We nevertheless present the details, since the proof for the general case in section 4 is similar. Care has to be taken of the gradings in tensor product multiplications and also in extending the antipode to the whole algebra:

$$\begin{aligned}
\Delta\left(\frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}} k_2^+(z)k_2^-(w)\right) &= \frac{wq^{-\frac{1}{2}(c_1+c_2)+1} - zq^{\frac{1}{2}(c_1+c_2)-1}}{zq^{\frac{1}{2}(c_1+c_2)+1} - wq^{-\frac{1}{2}(c_1+c_2)-1}} \left(k_2^+(zq^{\frac{1}{2}c_2}) \otimes k_2^+(zq^{-\frac{1}{2}c_1})\right) \\
&\quad \times \left(k_2^-(wq^{-\frac{1}{2}c_2}) \otimes k_2^-(wq^{\frac{1}{2}c_1})\right) \\
&= \frac{wq^{\frac{1}{2}(c_1+c_2)+1} - zq^{-\frac{1}{2}(c_1+c_2)-1}}{zq^{-\frac{1}{2}(c_1+c_2)+1} - wq^{\frac{1}{2}(c_1+c_2)-1}} \left(k_2^-(wq^{-\frac{1}{2}c_2}) \otimes k_2^-(wq^{\frac{1}{2}c_1})\right) \\
&\quad \times \left(k_2^+(zq^{\frac{1}{2}c_2}) \otimes k_2^+(zq^{-\frac{1}{2}c_1})\right) \\
&= \Delta\left(\frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}} k_2^-(w)k_2^+(z)\right)
\end{aligned} \tag{3.8}$$

where the relation between $k_2^+(z)$ and $k_2^-(w)$ has been used, and

$$\begin{aligned}
\Delta\left(\frac{z_+ - w_-}{z_+q - w_-q^{-1}} k_1^+(z)k_2^-(w)\right) &= \frac{zq^{\frac{1}{2}(c_1+c_2)} - wq^{-\frac{1}{2}(c_1+c_2)}}{zq^{\frac{1}{2}(c_1+c_2)+1} - wq^{-\frac{1}{2}(c_1+c_2)-1}} \\
&\quad \times \left(k_1^+(zq^{\frac{1}{2}c_2}) \otimes k_1^+(zq^{-\frac{1}{2}c_1})\right)\left(k_2^-(wq^{-\frac{1}{2}c_2}) \otimes k_2^-(wq^{\frac{1}{2}c_1})\right) \\
&= \frac{zq^{-\frac{1}{2}(c_1+c_2)} - wq^{\frac{1}{2}(c_1+c_2)}}{zq^{-\frac{1}{2}(c_1+c_2)+1} - wq^{\frac{1}{2}(c_1+c_2)-1}} \left(k_2^-(wq^{-\frac{1}{2}c_2}) \otimes k_2^-(wq^{\frac{1}{2}c_1})\right) \times
\end{aligned}$$

$$\times \left(k_1^+(zq^{\frac{1}{2}c_2}) \otimes k_1^+(zq^{-\frac{1}{2}c_1})\right)$$

$$= \Delta\left(\frac{z_- - w_+}{z_- q - w_+ q^{-1}} k_2^-(w) k_1^+(z)\right). \tag{3.9}$$

The relation between $\Delta(k_1^-(z))$ and $\Delta(k_2^+(w))$ can be proved along simliar lines.

The fifth and sixth equations in (3.3) are equivalent to the relations

$$\begin{aligned} &\phi_1(z) X_1^{\pm}(w) \phi_1(z)^{-1} = X_1^{\pm}(w) \\ &\psi_1(z) X_1^{\pm}(w) \psi_1(z)^{-1} = X_1^{\pm}(w) \end{aligned} \tag{3.10}$$

which are easily seen to be preserved by the coproduct, thanks to the commutativity of $\phi_1(z)$ and $\psi_1(w)$.

$$\begin{aligned}
\Delta\left(\{X_1^+(z), X_1^+(w)\}\right) &= \Delta(X_1^+(z))\Delta(X_1^+(w)) + \Delta(X_1^+(w))\Delta(X_1^+(z)) \\
&= X_1^+(z)X_1^+(w) \otimes 1 + X_1^+(z)\psi_1(wq^{\frac{1}{2}c_1}) \otimes X_1^+(wq^{c_1}) \\
&\quad - \psi_1(zq^{\frac{1}{2}c_1})X_1^+(w) \otimes X_1^+(zq^{c_1}) + X_1^+(w)X_1^+(z) \otimes 1 \\
&\quad + X_1^+(w)\psi_1(zq^{\frac{1}{2}c_1}) \otimes X_1^+(zq^{c_1}) - \psi_1(wq^{\frac{1}{2}c_1})X_1^+(z) \otimes X_1^+(wq^{c_1}) \\
&\quad + \psi_1(zq^{\frac{1}{2}c_1})\psi_1(wq^{\frac{1}{2}c_1}) \otimes X_1^+(zq^{c_1})X_1^+(wq^{c_1}) \\
&\quad + \psi_1(wq^{\frac{1}{2}c_1})\psi_1(zq^{\frac{1}{2}c_1}) \otimes X_1^+(wq^{c_1})X_1^+(zq^{c_1}) \\
&= \{X_1^+(z), X_1^+(w)\} \otimes 1 + \psi_1(zq^{\frac{1}{2}c_1})\psi_1(wq^{\frac{1}{2}c_1}) \\
&\quad \otimes \{X_1^+(zq^{c_1}), X_1^+(wq^{c_1})\} = 0.
\end{aligned} \tag{3.11}$$

$\Delta(\{X_1^-(z), X_1^-(w)\}) = 0$ can be proved similarly.

$$\begin{aligned}
\Delta\left(\{X_1^+(z), X_1^-(w)\}\right) &= \Delta(X_1^+(z))\Delta(X_1^-(w)) + \Delta(X_1^-(w))\Delta(X_1^+(z)) \\
&= X_1^+(z) \otimes X_1^-(w) + X_1^+(z)X_1^-(wq^{c_2}) \otimes \phi_1(wq^{\frac{1}{2}c_2}) \\
&\quad + \psi_1(zq^{\frac{1}{2}c_1}) \otimes X_1^+(zq^{c_1})X_1^-(w) - X_1^+(z) \otimes X_1^-(w) \\
&\quad + \psi_1(zq^{\frac{1}{2}c_1}) \otimes X_1^-(w)X_1^+(zq^{c_1}) + X_1^-(wq^{c_2})X_1^+(z) \otimes \phi_1(wq^{\frac{1}{2}c_2}) \\
&\quad - \psi_1(zq^{\frac{1}{2}c_1})X_1^-(wq^{c_2}) \otimes X_1^+(zq^{c_1})\phi_1(wq^{\frac{1}{2}c_2}) \\
&\quad + X_1^-(wq^{c_2})\psi_1(zq^{\frac{1}{2}c_1})X_1^+(zq^{c_1}) \\
&= [X_1^+(z), X_1^-(wq^{c_2})] \otimes \phi_1(wq^{c_2}) + \psi_1(zq^{\frac{1}{2}c_1}) \otimes [X_1^+(zq^{c_1}), X_1^-(w)] \\
&= (q - q^{-1})\left(\delta\left(\frac{w}{z}q^{c_1+c_2}\right)\phi_1(wq^{\frac{1}{2}c_2+\frac{1}{2}(c_1+c_2)})\phi_1(wq^{-\frac{1}{2}c_1+\frac{1}{2}(c_1+c_2)})\right. \\
&\quad \left. - \delta\left(\frac{w}{z}q^{-c_1-c_2}\right)\psi_1(zq^{-\frac{1}{2}c_2+\frac{1}{2}(c_1+c_2)})\psi_1(wq^{\frac{1}{2}c_1+\frac{1}{2}(c_1+c_2)})\right) \\
&= (q - q^{-1})\Delta\left(\delta\left(\frac{w}{z}q^{c}\right)\phi_1(w_+) - \delta\left(\frac{w}{z}q^{-c}\right)\psi_1(z_+)\right).
\end{aligned} \tag{3.12}$$

We have therefore proved that the comultiplication is an algebra homomorphism.

$$
\begin{aligned}
S(\{X_1^+(z), X_1^-(w)\}) &= -S(X_1^-(w))S(X_1^+(z)) - S(X_1^+(z))S(X_1^-(w)) \\
&= -\psi_1(zq^{-\frac{1}{2}c})^{-1}\phi_1(wq^{-\frac{1}{2}c})^{-1}[X_1^+(zq^{-c}), X_1^-(wq^{-c})] \\
&= -\psi_1(zq^{-\frac{1}{2}c})^{-1}\phi_1(wq^{-\frac{1}{2}c})^{-1}(q-q^{-1}) \\
&\quad \times \left(\delta\left(\frac{w}{z}q^c\right)\phi_1(w^{-\frac{1}{2}c}) - \delta\left(\frac{w}{z}q^{-c}\right)\psi_1(zq^{-\frac{1}{2}c})\right) \\
&= (q-q^{-1})\left(\delta\left(\frac{w}{z}q^{-c}\right)\phi_1(w^{-\frac{1}{2}c})^{-1} - \delta\left(\frac{w}{z}q^c\right)\psi_1(zq^{-\frac{1}{2}c})^{-1}\right) \\
&= (q-q^{-1})S\left(\delta\left(\frac{w}{z}q^c\right)\phi_1(w^{\frac{1}{2}c}) - \delta\left(\frac{w}{z}q^{-c}\right)\psi_1(zq^{\frac{1}{2}c})\right). \qquad (3.13)
\end{aligned}
$$

We can prove in the same manner that other relations are also preserved by the antipode.

Let $M: U_q[gl(1|1)^{(1)}] \otimes U_q[gl(1|1)^{(1)}] \rightarrow U_q[gl(1|1)^{(1)}]$ be a multiplication. Then we can easily check

$$
\begin{aligned}
&M(1\otimes\epsilon)\Delta = id = M(\epsilon\otimes 1)\Delta \\
&M(1\otimes S)\Delta = \epsilon = M(S\otimes 1)\Delta.
\end{aligned} \qquad (3.14)
$$

Thus we have shown that the coproduct, the counit and the antipode give a Hopf algebra structure.

4. The general case: $U_q[gl(m|n)^{(1)}]$

The generalization to the general case $U_q[gl(m|n)^{(1)}]$ is more-or-less straightforward using equations (2.15), (2.14), (2.18)–(2.24) and theorem 1. As in the non-supersymmetric cases [3, 4], this is achieved by induction on m and n. Tedious but direct computation gives rise to the following definition.

Definition 3. $U_q[gl(m|n)^{(1)}]$ is an associative algebra with unit 1 and Drinfeld current generators $X_i^{\pm}(z)$, $k_j^{\pm}(z)$, $i = 1, 2, \ldots, m+n-1$, $j = 1, 2, \ldots, m+n$, a central element c and a non-zero complex parameter q. $k_i^{\pm}(z)$ are invertible. The grading of the generators are: $[X_m^{\pm}(z)] = 1$ and zero otherwise. The defining relations are given by

$$
k_i^{\pm}(z)k_j^{\pm}(w) = k_j^{\pm}(w)k_i^{\pm}(z) \qquad i \neq j
$$

$$
k_i^{+}(z)k_i^{-}(w) = k_i^{-}(w)k_i^{+}(z) \qquad i \leqslant m
$$

$$
\frac{w_-q - z_+q^{-1}}{z_+q - w_-q^{-1}}k_i^{+}(z)k_i^{-}(w) = \frac{w_+q - z_-q^{-1}}{z_-q - w_+q^{-1}}k_i^{-}(w)k_i^{+}(z) \qquad m < i \leqslant m+n
$$

$$
\frac{z_\pm - w_\mp}{z_\pm q - w_\mp q^{-1}}k_i^{\mp}(w)^{-1}k_j^{\pm}(z) = \frac{z_\mp - w_\pm}{z_\mp q - w_\pm q^{-1}}k_j^{\pm}(z)k_i^{\mp}(w)^{-1} \qquad i > j
$$

$$
\left.
\begin{aligned}
&k_j^{\pm}(z)^{-1}X_i^{-}(w)k_j^{\pm}(z) = X_i^{-}(w) \qquad j-i \leqslant -1 \\
&k_j^{\pm}(z)^{-1}X_i^{+}(w)k_j^{\pm}(z) = X_i^{+}(w) \qquad j-i \leqslant -1
\end{aligned}
\right\}
$$

or
$$\begin{cases} k_j^{\pm}(z)^{-1}X_i^-(w)k_j^{\pm}(z) = X_i^-(w) & j-i \geqslant 2 \\ k_j^{\pm}(z)^{-1}X_i^+(w)k_j^{\pm}(z) = X_i^+(w) & j-i \geqslant 2 \end{cases}$$

$$k_i^{\pm}(z)^{-1}X_i^-(w)k_i^{\pm}(z) = \frac{z_{\mp}q - wq^{-1}}{z_{\mp} - w}X_i^-(w) \qquad i < m$$

$$k_i^{\pm}(z)^{-1}X_i^-(w)k_i^{\pm}(z) = \frac{z_{\mp}q^{-1} - wq}{z_{\mp} - w}X_i^-(w) \qquad m < i \leqslant m+n-1$$

$$k_{i+1}^{\pm}(z)^{-1}X_i^-(w)k_{i+1}^{\pm}(z) = \frac{z_{\mp}q^{-1} - wq}{z_{\mp} - w}X_i^-(w) \qquad i < m$$

$$k_{i+1}^{\pm}(z)^{-1}X_i^-(w)k_{i+1}^{\pm}(z) = \frac{z_{\mp}q - wq^{-1}}{z_{\mp} - w}X_i^-(w) \qquad m < i \leqslant m+n-1$$

$$k_i^{\pm}(z)X_i^+(w)k_i^{\pm}(z)^{-1} = \frac{z_{\pm}q - wq^{-1}}{z_{\pm} - w}X_i^+(w) \qquad i < m$$

$$k_i^{\pm}(z)X_i^+(w)k_i^{\pm}(z)^{-1} = \frac{z_{\pm}q^{-1} - wq}{z_{\pm} - w}X_i^+(w) \qquad m < i \leqslant m+n-1$$

$$k_{i+1}^{\pm}(z)X_i^+(w)k_{i+1}^{\pm}(z)^{-1} = \frac{z_{\pm}q^{-1} - wq}{z_{\pm} - w}X_i^+(w) \qquad i < m$$

$$k_{i+1}^{\pm}(z)X_i^+(w)k_{i+1}^{\pm}(z)^{-1} = \frac{z_{\pm}q - wq^{-1}}{z_{\pm} - w}X_i^+(w) \qquad m < i \leqslant m+n-1$$

$$k_i^{\pm}(z)^{-1}X_m^-(w)k_i^{\pm}(z) = \frac{z_{\mp}q - wq^{-1}}{z_{\mp} - w}X_m^-(w) \qquad i = m,\ m+1$$

$$k_i^{\pm}(z)X_m^+(w)k_i^{\pm}(z)^{-1} = \frac{z_{\pm}q - wq^{-1}}{z_{\pm} - w}X_m^+(w) \qquad i = m,\ m+1$$

$$(zq^{\mp1} - wq^{\pm1})X_i^{\mp}(z)X_i^{\mp}(w) = (zq^{\pm1} - wq^{\mp1})X_i^{\mp}(w)X_i^{\mp}(z) \qquad i < m$$

$$(wq^{\mp1} - zq^{\pm1})X_i^{\mp}(z)X_i^{\mp}(w) = (wq^{\pm1} - zq^{\mp1})X_i^{\mp}(w)X_i^{\mp}(z) \qquad m < i \leqslant m+n-1$$

$$\{X_m^{\pm}(z), X_m^{\pm}(w)\} = 0$$

$$(z-w)X_i^+(z)X_{i+1}^+(w) = (zq - wq^{-1})X_{i+1}^+(w)X_i^+(z) \qquad i < m$$

$$(w-z)X_i^+(z)X_{i+1}^+(w) = (wq - zq^{-1})X_{i+1}^+(w)X_i^+(z) \qquad m \leqslant i \leqslant m+n-1$$

$$(zq - wq^{-1})X_i^-(z)X_{i+1}^-(w) = (z-w)X_{i+1}^-(w)X_i^-(z) \qquad i < m$$

$$(wq - zq^{-1})X_i^-(z)X_{i+1}^-(w) = (w-z)X_{i+1}^-(w)X_i^-(z) \qquad m \leqslant i \leqslant m+n-1$$

$$[X_i^+(z), X_j^-(w)] = -(q-q^{-1})\delta_{ij}\left(\delta\left(\frac{w}{z}q^c\right)k_{i+1}^+(w_+)k_i^+(w_+)^{-1}\right.$$
$$\left. - \delta\left(\frac{w}{z}q^{-c}\right)k_{i+1}^-(z_+)k_i^-(z_+)^{-1}\right) \qquad i, j \neq m$$

$$\{X_m^+(z), X_m^-(w)\} = (q - q^{-1})\left(\delta\left(\frac{w}{z}q^c\right)k_{m+1}^+(w_+)k_m^+(w_+)^{-1} - \delta\left(\frac{w}{z}q^{-c}\right)k_{m+1}^-(z_+)k_m^-(z_+)^{-1}\right) \tag{4.1}$$

where $[X, Y] \equiv XY - YX$ stands for a commutator and $\{X, Y\} \equiv XY + YX$ for an anti-commutator, together with the following Serre and extra Serre [7, 8] relations:

$$\{X_i^\pm(z_1)X_i^\pm(z_2)X_{i+1}^\pm(w) - (q + q^{-1})X_i^\pm(z_1)X_{i+1}^\pm(w)X_i^\pm(z_2) + X_{i+1}^\pm(w)X_i^\pm(z_1)X_i^\pm(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \qquad i \neq m \tag{4.2}$$

$$\{X_{i+1}^\pm(z_1)X_{i+1}^\pm(z_2)X_i^\pm(w) - (q + q^{-1})X_{i+1}^\pm(z_1)X_i^\pm(w)X_{i+1}^\pm(z_2) + X_i^\pm(w)X_{i+1}^\pm(z_1)X_{i+1}^\pm(z_2)\} + \{z_1 \leftrightarrow z_2\} = 0 \qquad i \neq m-1 \tag{4.3}$$

$$\{(z_1q^{\mp 1} - z_2q^{\pm 1})[X_m^\pm(z_1)X_m^\pm(z_2)X_{m-1}^\pm(w) - (q + q^{-1})X_m^\pm(z_1)X_{m-1}^\pm(w)X_m^\pm(z_2) + X_{m-1}^\pm(w)X_m^\pm(z_1)X_m^\pm(z_2)]\} + \{z_1 \leftrightarrow z_2\} = 0 \tag{4.4}$$

$$\{(z_2q^{\mp 1} - z_1q^{\pm 1})[X_m^\pm(z_1)X_m^\pm(z_2)X_{m+1}^\pm(w) - (q + q^{-1})X_m^\pm(z_1)X_{m+1}^\pm(w)X_m^\pm(z_2) + X_{m+1}^\pm(w)X_m^\pm(z_1)X_m^\pm(z_2)]\} + \{z_1 \leftrightarrow z_2\} = 0 \tag{4.5}$$

$$\begin{aligned}\{(z_1q^{\mp 1} - z_2q^{\pm 1})&[X_m^\pm(z_1)X_m^\pm(z_2)X_{m-1}^\pm(w_1)X_{m+1}^\pm(w_2) \\ &- (q + q^{-1})X_m^\pm(z_1)X_{m-1}^\pm(w_1)X_m^\pm(z_2)X_{m+1}^\pm(w_2)] \\ &+ (z_1 + z_2)(q^{\mp 1} - q^{\pm 1})X_{m-1}^\pm(w_1)X_m^\pm(z_1)X_m^\pm(z_2)X_{m+1}^\pm(w_2) \\ &+ (z_2q^{\mp 1} - z_1q^{\pm 1})[-(q + q^{-1})X_{m-1}^\pm(w_1)X_m^\pm(z_1)X_{m+1}^\pm(w_2)X_m^\pm(z_2) \\ &+ X_{m-1}^\pm(w_1)X_{m+1}^\pm(w_2)X_m^\pm(z_1)X_m^\pm(z_2)]\} + \{z_1 \leftrightarrow z_2\} = 0.\end{aligned} \tag{4.6}$$

Theorem 3. The algebra $U_q[gl(m|n)^{(1)}]$ given by definition 3 has a Hopf algebra structure, which is given by the following formulae.

Coproduct Δ:

$$\begin{aligned}&\Delta(q^c) = q^c \otimes q^c \\ &\Delta(k_j^+(z)) = k_j^+(zq^{\frac{1}{2}c_2}) \otimes k_j^+(zq^{-\frac{c_1}{2}}) \\ &\Delta(k_j^-(z)) = k_j^-(zq^{-\frac{1}{2}c_2}) \otimes k_j^-(zq^{\frac{c_1}{2}}) \qquad j = 1, 2, \ldots, m+n \\ &\Delta(X_i^+(z)) = X_i^+(z) \otimes 1 + \psi_i(zq^{\frac{1}{2}c_1}) \otimes X_i^+(zq^{c_1}) \\ &\Delta(X_i^-(z)) = 1 \otimes X_i^-(z) + X_i^-(zq^{c_2}) \otimes \phi_i(zq^{\frac{1}{2}c_2}) \qquad i = 1, 2, \ldots, m+n-1\end{aligned} \tag{4.7}$$

where $c_1 = c \otimes 1$, $c_2 = 1 \otimes c$, $\psi_i(z) = k_{i+1}^-(z)k_i^-(z)^{-1}$ and $\phi_i(z) = k_{i+1}^+(z)k_i^+(z)^{-1}$.

Counit ϵ:

$$\epsilon(q^c) = 1 \qquad \epsilon(k_j^\pm(z)) = 1 \qquad \epsilon(X_i^\pm(z)) = 0. \tag{4.8}$$

Antipode S:

$$
\begin{aligned}
&S(q^c) = q^{-c} \qquad S(k_j^{\pm}(z)) = k_j^{\pm}(z)^{-1} \\
&S(X_i^+(z)) = -\psi_i(zq^{-\frac{1}{2}c})^{-1} X_i^+(zq^{-c}) \\
&S(X_i^-(z)) = -X_i^-(zq^{-c})\phi_i(zq^{-\frac{c}{2}})^{-1}.
\end{aligned}
\tag{4.9}
$$

Proof. This theorem is proved by direct calculation in a manner similar to that given in section 3 for $U_q[gl(1|1)^{(1)}]$.

Acknowledgments

This work is supported by Australian Research Council, and in part by University of Queensland New Staff Research Grant and External Support Enabling Grant.

Note added. After this paper appeared on the q-alg bulletin board, the work of Cai *et al* [9] appeared, where the authors obtained similar results for the simplest superalgebra $U_q[\widehat{gl(1|1)}]$.

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